

INFINITE MULTIPLICITY OF ABELIAN SUBALGEBRAS IN FREE GROUP SUBFACTORS

MARIUS B. ȘTEFAN

ABSTRACT. We obtain an estimate of Voiculescu's (modified) free entropy dimension for generators of a \mathbb{I}_1 -factor \mathcal{M} with a subfactor \mathcal{N} containing an abelian subalgebra \mathcal{A} of finite multiplicity. It implies in particular that the interpolated free group subfactors of finite Jones index do not have abelian subalgebras of finite multiplicity or Cartan subalgebras.

1. INTRODUCTION

Cartan subalgebras arise naturally in the classical group measure space construction. Thus, if α is a free action of a discrete countable group Γ on a measure space (X, μ) , then the cross-product von Neumann algebra $L^\infty(X, \mu) \rtimes_\alpha \Gamma$ contains a copy of $L^\infty(X, \mu)$ as a Cartan subalgebra. More generally, a Cartan subalgebra of a von Neumann algebra \mathcal{P} is a maximal abelian $*$ -subalgebra of \mathcal{P} whose normalizer generates \mathcal{P} (regular MASA) and which is the range of a normal conditional expectation ([Di], [FeMo]). D. Voiculescu defined ([Vo2], [Vo3]) an original concept of (modified) free entropy dimension δ_0 and proved ([Vo3]) that δ_0 of any finite system of generators of a von Neumann algebra which has a regular diffuse hyperfinite $*$ -subalgebra (regular DHSA) is ≤ 1 . This answered in the negative the longstanding open question of whether every separable \mathbb{I}_1 -factor contains a Cartan subalgebra since the free group factors $\mathcal{L}(\mathbb{F}_n)$ (von Neumann algebras generated by the left regular representations $\lambda : \mathbb{F}_n \rightarrow \mathcal{B}(l^2(\mathbb{F}_n))$, $2 \leq n \leq \infty$) have systems of generators with $\delta_0 > 1$. D. Voiculescu's result about the absence of Cartan subalgebras in free group factors was extended by L. Ge ([Ge]) and K. Dykema ([Dy2]) who showed that these factors do not have abelian subalgebras of multiplicity one and of finite multiplicity, respectively. We mention that if \mathcal{A} is a Cartan subalgebra in a \mathbb{I}_1 -factor N , then $(\mathcal{A} \cup J\mathcal{A}J)''$ is a MASA in $\mathcal{B}(L^2(\mathcal{N}, \tau))$ ([FeMo], [Po]), hence \mathcal{A} is in particular an abelian subalgebra of multiplicity one.

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The interpolated free group factors $\mathcal{L}(\mathbb{F}_t)$ ($1 < t \leq \infty$) were introduced independently by K. Dykema ([Dy2]) and F. Rădulescu ([Ră]) as a continuation of the discrete series $\mathcal{L}(\mathbb{F}_n)$, $2 \leq n \leq \infty$. We prove (Corollary 3.1) that the subfactors of finite Jones index in the interpolated free group factors do not have abelian subalgebras of finite multiplicity either. The result is a consequence of the estimate of (modified) free entropy dimension (Theorem 3.2) $\delta_0(x_1, \dots, x_m) \leq 2r + 2v + 3$, where x_1, \dots, x_m are self-adjoint generators of the \mathbb{I}_1 -factor \mathcal{M} , r is the integer part of the Jones index of \mathcal{N} in \mathcal{M} and v is the multiplicity of an abelian subalgebra \mathcal{A} in \mathcal{N} .

Schreier's Theorem describes all subgroups of finite index k in the free group \mathbb{F}_n : any such subgroup is isomorphic to the free group $\mathbb{F}_{1+k(n-1)}$. A von Neumann algebra analogue of the fact that $\mathbb{F}_{1+k(n-1)}$ can be embedded with finite index k in \mathbb{F}_n was proved by F. Rădulescu ([Ră]): $\mathcal{L}(\mathbb{F}_{1+\lambda^{-1}(t-1)})$ can be embedded in $\mathcal{L}(\mathbb{F}_t)$ with finite index $\lambda^{-1} \forall 1 < t \leq \infty \forall \lambda^{-1} \in \{4 \cos^2 \frac{\pi}{k} : k \geq 3\}$. On the other hand, at the von Neumann algebra level, with $\mathcal{L}(\mathbb{F}_n)$ instead of \mathbb{F}_n , it is no longer known whether Schreier's Theorem is still true. However, two properties are preserved when passing to free group subfactors of finite index: Haagerup approximation property ([Ha]) and primeness ([Št]) i.e., the indecomposability as tensor product of type \mathbb{I}_1 -factors. Our result about the absence of abelian subalgebras of finite multiplicity (and thus, of Cartan subalgebras) is a third property that seems to support the Schreier conjecture for free group subfactors.

We recall some results from D. Voiculescu's free probability theory ([Vo1], [Vo2], [Vo3]) for the reader's convenience. If \mathcal{M} is a \mathbb{I}_1 -factor with its unique faithful normalized trace τ then $\|x\|_s = \tau((x^*x)^{\frac{s}{2}})^{\frac{1}{s}}$ ($1 < s < \infty$) denotes the s -norm of $x \in \mathcal{M}$, $L^2(\mathcal{M}, \tau)$ denotes the completion of \mathcal{M} with respect to the 2-norm, and $\mathcal{M} \subset \mathcal{B}(L^2(\mathcal{M}, \tau))$ is the standard representation of \mathcal{M} . For an integer $c \geq 1$ let $\mathcal{M}_c(\mathbb{C})$ and $\mathcal{M}_c^{sa}(\mathbb{C})$ be the set of all $c \times c$ complex matrices and respectively, of all $c \times c$ complex self-adjoint matrices. Let further $\mathcal{U}_c(\mathbb{C})$ be the unitary group of $\mathcal{M}_c(\mathbb{C})$, τ_c be the unique normalized trace on $\mathcal{M}_c(\mathbb{C})$, and $\|\cdot\|_e = \sqrt{c}\|\cdot\|_2$ be the euclidian norm on $\mathcal{M}_c(\mathbb{C})$. The free entropy of $x_1, \dots, x_m \in \mathcal{M}^{sa}$ in the presence of $x_{m+1}, \dots, x_{m+n} \in \mathcal{M}^{sa}$ is defined in terms of sets of matricial microstates $\Gamma_R((x_i)_{1 \leq i \leq m} : (x_{m+j})_{1 \leq j \leq n}; p, c, \epsilon) \subset (\mathcal{M}_c^{sa}(\mathbb{C}))^m$. The set Γ_R of matricial microstates corresponding to integers $c, p \geq 1$ and to $\epsilon > 0$ consists in m -tuples $(A_i)_{1 \leq i \leq m}$ of $c \times c$ self-adjoint matrices such that there exists an n -tuple $(A_{m+j})_{1 \leq j \leq n} \in (\mathcal{M}_c^{sa}(\mathbb{C}))^n$ with the properties

$$|\tau(x_{i_1} \dots x_{i_l}) - \tau_c(A_{i_1} \dots A_{i_l})| < \epsilon, \quad \|A_k\| \leq R$$

for all $1 \leq i_1, \dots, i_l \leq m+n$, $1 \leq l \leq p$, $1 \leq k \leq m+n$. One defines then successively:

$$\begin{aligned}
(1) \quad & \chi_R((x_i)_{1 \leq i \leq m} : (x_{m+j})_{1 \leq j \leq n}; p, c, \epsilon) \\
& = \log \text{vol}_{mc^2}(\Gamma_R((x_i)_{1 \leq i \leq m} : (x_{m+j})_{1 \leq j \leq n}; p, c, \epsilon)), \\
(2) \quad & \chi_R((x_i)_{1 \leq i \leq m} : (x_{m+j})_{1 \leq j \leq n}; p, \epsilon) \\
& = \limsup_{c \rightarrow \infty} \left(\frac{1}{c^2} \chi_R((x_i)_{1 \leq i \leq m} : (x_{m+j})_{1 \leq j \leq n}; p, c, \epsilon) + \frac{m}{2} \log c \right), \\
(3) \quad & \chi_R((x_i)_{1 \leq i \leq m} : (x_{m+j})_{1 \leq j \leq n}) \\
& = \inf_{p, \epsilon} \chi_R((x_i)_{1 \leq i \leq m} : (x_{m+j})_{1 \leq j \leq n}; p, \epsilon), \\
(4) \quad & \chi((x_i)_{1 \leq i \leq m} : (x_{m+j})_{1 \leq j \leq n}) \\
& = \sup_R \chi_R((x_i)_{1 \leq i \leq m} : (x_{m+j})_{1 \leq j \leq n})
\end{aligned}$$

(we denoted by $\text{vol}_{mc^2}(\cdot)$ the Lebesgue measure on $(\mathcal{M}_c^{sa}(\mathbb{C}))^m \simeq \mathbb{R}^{mc^2}$). The resulting quantity $\chi((x_i)_{1 \leq i \leq m} : (x_{m+j})_{1 \leq j \leq n})$ is the free entropy of $(x_i)_{1 \leq i \leq m}$ in the presence of $(x_{m+j})_{1 \leq j \leq n}$ or if $n = 0$, the free entropy $\chi(x_1, \dots, x_m)$ of $(x_i)_{1 \leq i \leq m}$. The free entropy of $(x_i)_{1 \leq i \leq m}$ in the presence of $(x_{m+j})_{1 \leq j \leq n}$ is equal to the free entropy of $(x_i)_{1 \leq i \leq m}$ if $\{x_{m+1}, \dots, x_{m+n}\} \subset \{x_1, \dots, x_m\}''$. Also, the free entropy of a single self-adjoint element x is (where μ denotes the distribution of x):

$$\chi(x) = \frac{3}{4} + \frac{1}{2} \log 2\pi + \int \int \log |s - t| d\mu(s) d\mu(t).$$

An element $x \in \mathcal{M}$ is a semicircular element if it is self-adjoint and if its distribution is given by the semicircle law:

$$\tau(x^k) = \frac{2}{\pi} \int_{-1}^1 t^k \sqrt{1 - t^2} dt \quad \forall k \in \mathbb{N}.$$

A family $(\mathcal{M}_i)_{i \in I}$ of unital $*$ -subalgebras of \mathcal{M} is a free family if $\tau(x_k) = 0$, $x_k \in \mathcal{M}_{i_k} \quad \forall 1 \leq k \leq p$, $i_1, \dots, i_p \in I$, $i_1 \neq i_2 \neq \dots \neq i_p$, $p \in \mathbb{N}$ imply $\tau(x_1 \dots x_p) = 0$. A family $(A_i)_{i \in I}$ of subsets $A_i \subset \mathcal{M}$ is free if the family $(\ast\text{-alg}(\{1\} \cup A_i))_{i \in I}$ is free. A free set $(s_i)_{1 \leq i \leq m} \subset \mathcal{M}$ consisting of semicircular elements is called a semicircular system. If $(x_i)_{1 \leq i \leq m}$ is free then $\chi(x_1, \dots, x_m) = \chi(x_1) + \dots + \chi(x_m)$ hence a finite semicircular system has finite free entropy. The modified free entropy dimension and the free entropy dimension of an m -tuple of self-adjoint elements $(x_i)_{1 \leq i \leq m} \subset \mathcal{M}$ are

$$\delta_0((x_i)_{1 \leq i \leq m}) = m + \limsup_{\omega \rightarrow 0} \frac{\chi((x_i + \omega s_i)_{1 \leq i \leq m} : (s_i)_{1 \leq i \leq m})}{|\log \omega|}$$

and respectively,

$$\delta((x_i)_{1 \leq i \leq m}) = m + \limsup_{\omega \rightarrow 0} \frac{\chi((x_i + \omega s_i)_{1 \leq i \leq m})}{|\log \omega|},$$

where $(x_i)_{1 \leq i \leq m}$ and the semicircular system $(s_i)_{1 \leq i \leq m}$ are free. If x_1, \dots, x_m are free, then

$$\delta_0((x_i)_{1 \leq i \leq m}) = \delta((x_i)_{1 \leq i \leq m}) = \sum_{i=1}^m \delta(x_i).$$

Moreover, for a single self-adjoint element $x \in \mathcal{M}$ one has

$$\delta(x) = 1 - \sum_{s \in \mathbb{R}} (\mu(\{s\}))^2,$$

therefore $\delta(x) = 1$ if the distribution of x has no atoms.

2. ESTIMATE OF FREE ENTROPY

We obtain an estimate of the free entropy $\chi(x_1, \dots, x_m)$ for self-adjoint elements x_1, \dots, x_m which can be approximated in the $\|\cdot\|_2$ -norm by certain noncommutative polynomials of degree 1 in some of their variables. The proof of Lemma 2.1 is based on the observation that in this case the $c \times c$ matricial microstates of x_1, \dots, x_m are concentrated in some neighborhood of a linear subspace in $\mathcal{M}_c^{sa}(\mathbb{C})$.

Lemma 2.1. *Let x_1, \dots, x_m be self-adjoint elements that generate a \mathcal{H}_1 -factor (\mathcal{M}, τ) . Assume that there exist self-adjoint elements $m_j^{(l)}, z_k \in \mathcal{M}$ (for $1 \leq j \leq r+1$, $1 \leq l \leq 2$, $1 \leq k \leq 2v$), mutually orthogonal projections $p_q \in \mathcal{M}$ (for $1 \leq q \leq u$), noncommutative polynomials $\Phi_{ji}^{(l)}((p_q)_q, (z_k)_k) = \sum_{k=1}^{2v} \sum_{q,s=1}^u \mu_{q,s}^{(i,j,k,l)} p_q z_k p_s$ (where $\mu_{q,s}^{(i,j,k,l)}$ are scalars), and $0 < \omega < \frac{1}{3}$ such that*

$$\left\| x_i - \frac{1}{2} \sum_{l=1}^2 \sum_{j=1}^{r+1} (m_j^{(l)} \Phi_{ji}^{(l)}((p_q)_q, (z_k)_k) + \Phi_{ji}^{(l)}((p_q)_q, (z_k)_k)^* m_j^{(l)}) \right\|_2 < \omega$$

for all $1 \leq i \leq m$. Then

$$(5) \quad \chi(x_1, \dots, x_m) \leq C(m, r, v, K) + (m - 2r - 2v - 3) \log \omega,$$

where $C(m, r, v, K)$ is a constant depending only on m, r, v , and $K = 1 + \max_{i,j,l} \{\|\Phi_{ji}^{(l)}((p_q)_q, (z_k)_k)\|_2, \|x_i\|, \|m_j^{(l)}\|\}$.

Proof. For $R, \frac{1}{\epsilon} > 0$ sufficiently large and integer $p \geq 1$ consider $(A_1, \dots, A_m, (M_j^{(l)})_{j,l}, (P_q)_q, (Z_k)_k)$, an arbitrary element of the set of matricial microstates $\Gamma_R(x_1, \dots, x_m, (m_j^{(l)})_{j,l}, (p_q)_q, (z_k)_k; p, c, \epsilon)$. One

can assume (see [Vo3]) that $\|A_i\|, \|M_j^{(l)}\|, \|P_q\| \leq K$. If p is large and $\epsilon > 0$ is small enough, then

$$(6) \quad \left\| A_i - \frac{1}{2} \sum_{l=1}^2 \sum_{j=1}^{r+1} (M_j^{(l)} \Phi_{ji}^{(l)}((P_q)_q, (Z_k)_k) + \Phi_{ji}^{(l)}((P_q)_q, (Z_k)_k)^* M_j^{(l)}) \right\|_2 < \omega$$

for all $1 \leq i \leq m$ and $\|\Phi_{ji}^{(l)}((P_q)_q, (Z_k)_k)\|_2 < K$ for all i, j, l . Lemma 4.3 in [Vo2] implies that for any $\delta > 0$ there exist $p', c' \in \mathbb{N}, \epsilon_1 > 0$ such that if $c \geq c'$ and if $(P_1, \dots, P_u) \in \Gamma_R((p_q)_q; p', c, \epsilon_1)$, then there exist mutually orthogonal projections $Q_1, \dots, Q_u \in \mathcal{M}_c^{sa}(\mathbb{C})$ such that $\text{rank}(Q_q) = [\tau(p_q)c]$ and $\|P_q - Q_q\|_2 < \delta \forall 1 \leq q \leq u$. If $\delta > 0$ is sufficiently small one has then for all $c \geq c'$ and for all $1 \leq i \leq m$,

$$(7) \quad \left\| A_i - \frac{1}{2} \sum_{l=1}^2 \sum_{j=1}^{r+1} (M_j^{(l)} \Phi_{ji}^{(l)}((Q_q)_q, (Z_k)_k) + \Phi_{ji}^{(l)}((Q_q)_q, (Z_k)_k)^* M_j^{(l)}) \right\|_2 < \omega$$

and $\|\Phi_{ji}^{(l)}((Q_q)_q, (Z_k)_k)\|_2 < K$ for all i, j, l . Let $S_1, \dots, S_u \in \mathcal{M}_c^{sa}(\mathbb{C})$ be mutually orthogonal projections, fixed, with each projection S_q of rank $[\tau(p_q)c]$. There exists then $U \in \mathcal{U}_c(\mathbb{C})$ such that $Q_q = U^* S_q U$ for all $1 \leq q \leq u$ and one obtains

$$(8) \quad \left\| U A_i U^* - \frac{1}{2} \sum_{l=1}^2 \sum_{j=1}^{r+1} (B_j^{(l)} \Phi_{ji}^{(l)}((S_q)_q, (T_k)_k) + \Phi_{ji}^{(l)}((S_q)_q, (T_k)_k)^* B_j^{(l)}) \right\|_2 < \omega$$

for all $1 \leq i \leq m$, where we denoted $B_j^{(l)} = U M_j^{(l)} U^*$, $T_k = U Z_k U^*$. Let $\{U_a\}_{a \in A(c)}$ be a minimal γ -net in $\mathcal{U}_c(\mathbb{C})$ with respect to the $\|\cdot\|$ -norm. According to a result of S. J. Szarek ([Sz]), $|A(c)| \leq \left(\frac{C}{\gamma}\right)^{c^2}$ for some universal constant C . Consider also a minimal θ -net $\{V_b\}_{b \in B(c, K)}$ in $\{B \in \mathcal{M}_c^{sa}(\mathbb{C}) : \|B\| \leq K\}$, with respect to the same norm. It is easily seen that Szarek's result implies $|B(c, K)| \leq \left(\frac{CK}{\theta}\right)^{c^2+c}$. Since $\|U A_i U^* - U_a A_i U_a^*\|_2 < 2K\gamma$ for some $a \in A(c)$ and $\|B_j^{(l)} - V_{b(j,l)}\| < \theta$

for some $b(j, l) \in B(c, K)$, we have

$$\begin{aligned}
(9) \quad & \left\| U_a A_i U_a^* - \frac{1}{2} \sum_{l=1}^2 \sum_{j=1}^{r+1} (V_{b(j,l)} \Phi_{ji}^{(l)}((S_q)_q, (T_k)_k) + \Phi_{ji}^{(l)}((S_q)_q, \right. \\
& \left. (T_k)_k)^* V_{b(j,l)}) \right\|_2 \leq \|U_a A_i U_a^* - U_a A_i U_a^*\|_2 + \left\| U_a A_i U_a^* - \frac{1}{2} \right. \\
& \cdot \sum_{l=1}^2 \sum_{j=1}^{r+1} (B_j^{(l)} \Phi_{ji}^{(l)}((S_q)_q, (T_k)_k) + \Phi_{ji}^{(l)}((S_q)_q, (T_k)_k)^* B_j^{(l)}) \left\|_2 \right. \\
& + \sum_{l=1}^2 \sum_{j=1}^{r+1} \|B_j^{(l)} - V_{b(j,l)}\| \cdot \|\Phi_{ji}^{(l)}((S_q)_q, (T_k)_k)\|_2 \\
& < 2K\gamma + \omega + 2(r+1)K\theta = 3\omega \forall 1 \leq i \leq m.
\end{aligned}$$

Choose $\gamma = \frac{\omega}{2K}$, $\theta = \frac{\omega}{2(r+1)K}$, and define the function $F = (F_i((T_k)_k))_i : (\mathcal{M}_c^{sa}(\mathbb{C}))^{2v} \rightarrow (\mathcal{M}_c^{sa}(\mathbb{C}))^m$ by

$$\begin{aligned}
(10) \quad F_i((T_k)_k) &= \frac{1}{2} \sum_{l=1}^2 \sum_{j=1}^{r+1} U_a^* (V_{b(j,l)} \Phi_{ji}^{(l)}((S_q)_q, (T_k)_k) \\
&+ \Phi_{ji}^{(l)}((S_q)_q, (T_k)_k)^* V_{b(j,l)}) U_a \forall 1 \leq i \leq m.
\end{aligned}$$

It follows from (9) that the distance in the euclidian norm from the microstate (A_1, \dots, A_m) to the image of F is less than or equal to $3\omega\sqrt{mc}$. The polynomials $\Phi_{ji}^{(l)}$ are linear in $(T_k)_k$, hence the image of F is a linear subspace in $(\mathcal{M}_c^{sa}(\mathbb{C}))^m$, of dimension $d_F \leq 2vc^2$. Denote by $L_F(\omega, c)$ the intersection of this subspace with the ball of euclidian radius $(3\omega + K)\sqrt{mc}$ and by $B_F(\omega, c)$ the cartesian product of $L_F(\omega, c)$ with the ball of (euclidian) radius $3\omega\sqrt{mc}$ in the orthogonal complement of the image of F . The set of matricial microstates $\Gamma_R(x_1, \dots, x_m : (m_j^{(l)})_{j,l}, (p_q)_q, (z_k)_k; p, c, \epsilon)$ is contained in $\bigcup_F B_F(\omega, c)$,

hence

$$\begin{aligned}
(11) \quad & \text{vol}_{mc^2}(\Gamma_R(x_1, \dots, x_m : (m_j^{(l)})_{j,l}, (p_q)_q, (z_k)_k; p, c, \epsilon)) \\
& \leq \sum_F \text{vol}_{mc^2}(B_F(\omega, c)) \\
& = \sum_F \text{vol}_{d_F}((3\omega + K)\sqrt{mc}) \cdot \text{vol}_{mc^2-d_F}(3\omega\sqrt{mc}) \\
& = \sum_F \frac{(\pi mc)^{\frac{d_F}{2}} (3\omega + K)^{d_F}}{\Gamma(1 + \frac{d_F}{2})} \cdot \frac{(\pi mc)^{\frac{mc^2-d_F}{2}} (3\omega)^{mc^2-d_F}}{\Gamma(1 + \frac{mc^2-d_F}{2})} \\
& \leq \left(\frac{2CK}{\omega}\right)^{c^2} \cdot \left[\left(\frac{2(r+1)CK^2}{\omega}\right)^{c^2+c}\right]^{2(r+1)} \\
& \quad \cdot \frac{(\pi mc)^{\frac{mc^2}{2}} (2K)^{2vc^2} (3\omega)^{(m-2v)c^2} 2^{mc^2}}{\Gamma(1 + \frac{mc^2}{2})}.
\end{aligned}$$

After taking the limit after $c, p, \frac{1}{\epsilon} \rightarrow \infty$ in the resulting upper bound for $\chi_R(x_1, \dots, x_m : (m_j^{(l)})_{j,l}, (p_q)_q, (z_k)_k; p, c, \epsilon)$, eliminating R as in the definition of free entropy, and recalling that $\{x_1, \dots, x_m\}$ is a system of generators, one obtains

$$\begin{aligned}
(12) \quad \chi(x_1, \dots, x_m) &= \chi(x_1, \dots, x_m : (m_j^{(l)})_{j,l}, (p_q)_q, (z_k)_k) \\
&\leq C(m, r, v, K) + (m - 2r - 2v - 3) \log \omega.
\end{aligned}$$

□

3. INFINITE MULTIPLICITY

Let \mathcal{P} be a von Neumann algebra. If $\mathcal{Q} \subset \mathcal{P}$ is a subalgebra, then the normalizer of \mathcal{Q} in \mathcal{P} is by definition the set $N_{\mathcal{P}}(\mathcal{Q}) = \{u \in \mathcal{P} : uu^* = u^*u = 1, u\mathcal{Q}u^* = \mathcal{Q}\}$.

Definition 3.1. ([Di], [FeMo]) *A Cartan subalgebra of a von Neumann algebra \mathcal{P} is a maximal abelian $*$ -subalgebra (MASA) $\mathcal{A} \subset \mathcal{P}$ such that: i) \mathcal{A} is the range of a normal conditional expectation; ii) the normalizer $N_{\mathcal{P}}(\mathcal{A})$ of \mathcal{A} in \mathcal{P} generates \mathcal{P} .*

If \mathcal{N} is a type II_1 -factor, then the representation $\mathcal{N} \subset \mathcal{B}(L^2(\mathcal{N}, \tau))$ (τ denotes the unique normalized trace on \mathcal{N}) is the standard form of \mathcal{N} . Let $J : L^2(\mathcal{N}, \tau) \rightarrow L^2(\mathcal{N}, \tau)$ be the modular conjugacy operator. We recall the following Theorem due to J. Feldman and C. C. Moore:

Theorem 3.1. ([FeMo], [Po]) *Let \mathcal{N} be a type II_1 -factor. If \mathcal{A} is a Cartan subalgebra of \mathcal{N} , then the algebra $(\mathcal{A} \cup J\mathcal{A}J)''$ is maximal abelian in $\mathcal{B}(L^2(\mathcal{N}, \tau))$.*

Being a MASA, the algebra $(\mathcal{A} \cup J\mathcal{A}J)''$ has a cyclic vector $\xi \in L^2(\mathcal{N}, \tau)$ i.e., $\overline{\text{sp}}^{||\cdot||_2}(\mathcal{A} \cup J\mathcal{A}J)''\xi = L^2(\mathcal{N}, \tau)$. With the usual identification of ${}_J\mathcal{A}J L^2(\mathcal{N}, \tau)$ with $L^2(\mathcal{N}, \tau)_{\mathcal{A}}$, this means that $\overline{\text{sp}}^{||\cdot||_2}\mathcal{A}\xi\mathcal{A} = L^2(\mathcal{N}, \tau)$ that is, \mathcal{A} has finite multiplicity 1 in \mathcal{N} .

Definition 3.2. ([Dy2]) *An abelian subalgebra \mathcal{A} of a type II_1 -factor \mathcal{N} has finite multiplicity $\leq v < \infty$ if there exist v vectors $\xi_1, \dots, \xi_v \in L^2(\mathcal{N}, \tau)$ such that*

$$\overline{\text{sp}}^{||\cdot||_2}(\mathcal{A}\xi_1\mathcal{A} + \dots + \mathcal{A}\xi_v\mathcal{A}) = L^2(\mathcal{N}, \tau)$$

or equivalently, if ${}_A L^2(\mathcal{N}, \tau)_A$ is generated as an \mathcal{A} - \mathcal{A} -bimodule by v vectors from $L^2(\mathcal{N}, \tau)$. If ${}_A L^2(\mathcal{N}, \tau)_A$ is not a finitely generated \mathcal{A} - \mathcal{A} -bimodule, we say that \mathcal{A} has infinite multiplicity.

The multiplicity of \mathcal{A} in \mathcal{N} does not increase after compressing with a projection $p \in \mathcal{A}$:

Lemma 3.1. ([Dy2]) *If $\mathcal{A} \subset \mathcal{N}$ has finite multiplicity $\leq v$ and $p \in \mathcal{A}$ is an arbitrary projection, then $\mathcal{A}_p = p\mathcal{A} \subset p\mathcal{N}p = \mathcal{N}_p$ has also finite multiplicity $\leq v$.*

Theorem 3.2. *Let (\mathcal{M}, τ) be a II_1 -factor generated by the self-adjoint elements x_1, \dots, x_m . If $\mathcal{N} \subset \mathcal{M}$ is a subfactor with the integer part of the Jones index $[\mathcal{M} : \mathcal{N}]$ equal to r and if $\mathcal{A} \subset \mathcal{N}$ is an abelian subalgebra of multiplicity $\leq v$, then*

$$\delta_0(x_1, \dots, x_m) \leq 2r + 2v + 3.$$

Proof. We can assume from the beginning that $m > 2r + 2v + 3$ since $\delta_0(x_1, \dots, x_m) \leq m$ is always true ([Vo3]). There exists a Pimsner-Popa basis ([PiPo]) $m_1, \dots, m_{r+1} \in \mathcal{M}$ such that

$$x = \sum_{j=1}^{r+1} m_j E_{\mathcal{N}}(m_j^* x) \quad \forall x \in \mathcal{M},$$

where $E_{\mathcal{N}} : \mathcal{M} \rightarrow \mathcal{N}$ is the conditional expectation from \mathcal{M} onto \mathcal{N} . Denote the embedding $\mathcal{N} \subset L^2(\mathcal{N}, \tau)$ by $x \mapsto \hat{x}$ and let $J : L^2(\mathcal{N}, \tau) \rightarrow L^2(\mathcal{N}, \tau)$ be the modular conjugacy operator defined by $J(\hat{x}) = \widehat{x^*}$. Let $\xi_1, \dots, \xi_v \in L^2(\mathcal{N}, \tau)$ such that

$$\mathcal{A}\xi_1\mathcal{A} + \dots + \mathcal{A}\xi_v\mathcal{A}$$

is a dense subset of $L^2(\mathcal{N}, \tau)$. Eventually replacing ξ_i by $\frac{1}{2}(\xi_i + J\xi_i) + \frac{1}{2\sqrt{-1}}(\xi_i - J\xi_i)\sqrt{-1}$ and regrouping, we can assume that there exist

$\eta_1, \dots, \eta_{2v} \in L^2(\mathcal{N}, \tau)^{sa} := \{\xi \in L^2(\mathcal{N}, \tau) : J\xi = \xi\}$ such that $\mathcal{A}\eta_1\mathcal{A} + \dots + \mathcal{A}\eta_{2v}\mathcal{A}$ is dense in $L^2(\mathcal{N}, \tau)$. Let x_1, \dots, x_m be self-adjoint elements of \mathcal{M} . Every element $E_{\mathcal{N}}(m_j^*x_i) \in \mathcal{N}$ can be approximated arbitrarily well in the $\|\cdot\|_2$ -norm by elements of the form

$$\sum_{k=1}^{2v} \sum_{p=1}^t a_{p,k}^{(i,j)} \eta_k b_{p,k}^{(i,j)}$$

for some $a_{p,k}^{(i,j)}, b_{p,k}^{(i,j)} \in \mathcal{A}$. Since \mathcal{A} is abelian, there exist an integer u and projections p_1, \dots, p_u of sum 1 such that every $a_{p,k}^{(i,j)}$ and $b_{p,k}^{(i,j)}$ can be approximated sufficiently well in the uniform norm, by linear combinations of these projections. Moreover, $\widehat{\mathcal{N}^{sa}}$ is dense in $L^2(\mathcal{N}, \tau)^{sa}$ so one can find z_1, \dots, z_{2v} self-adjoint elements of \mathcal{N} and scalars $\mu_{q,s}^{(i,j,k)} \in \mathbb{C}$ such that

$$\Psi_{ji}((p_q)_q, (z_k)_k) = \sum_{k=1}^{2v} \sum_{q,s=1}^u \mu_{q,s}^{(i,j,k)} p_q z_k p_s$$

is sufficiently close to $E_{\mathcal{N}}(m_j^*x_i)$ in the $\|\cdot\|_2$ -norm, for all indices i, j . In particular, one can arrange for the norms $\|\Psi_{ji}((p_q)_q, (z_k)_k)\|_2$ to be all uniformly bounded by a constant D depending only on the norms $\|m_j^*x_i\|$. Therefore, every element x_i can be approximated arbitrarily well in the $\|\cdot\|_2$ -norm, by elements of the form

$$\sum_{j=1}^{r+1} m_j \Psi_{ji}((p_q)_q, (z_k)_k).$$

Denote $m_j^{(1)} = \frac{1}{2}(m_j + m_j^*)$ and $m_j^{(2)} = \frac{1}{2\sqrt{-1}}(m_j - m_j^*)$. It follows that every element x_i can be approximated arbitrarily well in the $\|\cdot\|_2$ -norm, by elements of the form

$$\sum_{l=1}^2 \sum_{j=1}^{r+1} m_j^{(l)} \Phi_{ji}^{(l)}((p_q)_q, (z_k)_k),$$

where $\Phi_{ji}^{(1)}((p_q)_q, (z_k)_k) = \Psi_{ji}((p_q)_q, (z_k)_k) = -\sqrt{-1}\Phi_{ji}^{(2)}((p_q)_q, (z_k)_k)$. Since $x_i = x_i^* \forall 1 \leq i \leq m$, given $\omega > 0$, every element x_i can ultimately be approximated in the $\|\cdot\|_2$ -norm as

$$\left\| x_i - \frac{1}{2} \sum_{l=1}^2 \sum_{j=1}^{r+1} (m_j^{(l)} \Phi_{ji}^{(l)}((p_q)_q, (z_k)_k) + \Phi_{ji}^{(l)}((p_q)_q, (z_k)_k)^* m_j^{(l)}) \right\|_2 < \omega.$$

If s_1, \dots, s_m is a semicircular system free from x_1, \dots, x_m then ([Vo3])

$$(13) \quad \chi((x_i + \omega s_i)_{1 \leq i \leq m} : (s_i)_{1 \leq i \leq m}) = \chi((x_i + \omega s_i)_{1 \leq i \leq m} : (s_i)_{1 \leq i \leq m}, (m_j^{(l)})_{j,l}, (p_q)_q, (z_k)_k) \leq \chi((x_i + \omega s_i)_{1 \leq i \leq m} : (m_j^{(l)})_{j,l}, (p_q)_q, (z_k)_k)$$

since $(m_j^{(l)})_{j,l}, (p_q)_q, (z_k)_k \subset \{x_i + \omega s_i, s_i \mid 1 \leq i \leq m\}''$. Note that

$$(14) \quad \left\| x_i + \omega s_i - \frac{1}{2} \sum_{l=1}^2 \sum_{j=1}^{r+1} (m_j^{(l)} \Phi_{ji}^{(l)}((p_q)_q, (z_k)_k) + \Phi_{ji}^{(l)}((p_q)_q, (z_k)_k)^* m_j^{(l)}) \right\|_2 < 2\omega$$

for all $1 \leq i \leq m$, hence the estimate of free entropy from Lemma 2.1 implies

$$\chi((x_i + \omega s_i)_{1 \leq i \leq m} : (s_i)_{1 \leq i \leq m}) \leq C(m, r, v, K) + (m - 2r - 2v - 3) \log 2\omega,$$

therefore

$$(15) \quad \begin{aligned} \delta_0(x_1, \dots, x_m) &= m + \limsup_{\omega \rightarrow 0} \frac{\chi((x_i + \omega s_i)_{1 \leq i \leq m} : (s_i)_{1 \leq i \leq m})}{|\log \omega|} \\ &\leq m + \limsup_{\omega \rightarrow 0} \frac{C(m, r, v, K) + (m - 2r - 2v - 3) \log 2\omega}{|\log \omega|} \\ &= m - (m - 2r - 2v - 3) = 2r + 2v + 3. \end{aligned}$$

□

Corollary 3.1. *The subfactors \mathcal{N} of finite index in the interpolated free group factors $\mathcal{L}(\mathbb{F}_t)$, $1 < t \leq \infty$, do not contain abelian subalgebras of finite multiplicity.*

Proof. Consider first the case $1 < t < \infty$ and suppose that \mathcal{N} has an abelian subalgebra \mathcal{A} of finite multiplicity $\leq v$. For every projection $p \in \mathcal{A}$, $p\mathcal{A}$ is an abelian subalgebra of multiplicity $\leq v$ in $p\mathcal{N}p$ (Lemma 3.1). Moreover ([Jo]), $[\mathcal{L}(\mathbb{F}_t)_p : p\mathcal{N}p] = [\mathcal{L}(\mathbb{F}_t) : \mathcal{N}] < \infty$. Eventually replacing \mathcal{A} by a MASA in \mathcal{N} , which contains \mathcal{A} (and thus, is of finite multiplicity $\leq v$ in \mathcal{N}), we can assume that \mathcal{A} is a MASA in \mathcal{N} , hence has no minimal projections. Therefore, there exists a projection $p \in \mathcal{A}$ such that $m = 1 + \frac{t-1}{\tau(p)^2}$ is a conveniently large integer (i.e., $m > 2r + 2v + 3$). Theorem 3.2 implies that the (modified) free entropy dimension of any finite system of generators of $\mathcal{L}(\mathbb{F}_t)_p \simeq \mathcal{L}(\mathbb{F}_m)$ (compression formula in [Dy1], [Ră]) is $\leq 2r + 2v + 3$, and this is in contradiction with the fact that $\mathcal{L}(\mathbb{F}_m)$ is generated by a semicircular system with $\delta_0 = m$ ([Vo2], [Vo3]).

Suppose now $t = \infty$ and let x_1, x_2, \dots be an infinite semicircular system that generates $\mathcal{L}(\mathbb{F}_\infty)$. Making use of inequality (12), one obtains:

$$\chi(x_1, \dots, x_m : (m_j^{(l)})_{j,l}, (p_q)_q, (z_k)_k) < \chi(x_1, \dots, x_m),$$

for some suitable elements. Let E_n , $n \geq 1$, be the conditional expectation from $\mathcal{L}(\mathbb{F}_\infty)$ onto $\{x_1, \dots, x_n\}''$. The convergence in distribution ([Vo2], [Vo3]) implies the existence of a integer $n > m$ such that

$$(16) \quad \chi\left(x_1, \dots, x_m : \left(E_n\left(m_j^{(l)}\right)\right)_{j,l}, (E_n(p_q))_q, (E_n(z_k))_k\right) < \chi(x_1, \dots, x_m).$$

One obtains then a contradiction:

$$(17) \quad \begin{aligned} \chi(x_1, \dots, x_n) &= \chi\left(x_1, \dots, x_n : \left(E_n\left(m_j^{(l)}\right)\right)_{j,l}, (E_n(p_q))_q, \right. \\ &\quad \left. (E_n(z_k))_k\right) \leq \chi\left(x_1, \dots, x_m : \left(E_n\left(m_j^{(l)}\right)\right)_{j,l}, (E_n(p_q))_q, \right. \\ &\quad \left. (E_n(z_k))_k\right) + \chi(x_{m+1}, \dots, x_n) < \chi(x_1, \dots, x_m) \\ &\quad + \chi(x_{m+1}, \dots, x_n) = \chi(x_1, \dots, x_n). \end{aligned}$$

□

Corollary 3.2. *The interpolated free group subfactors (of finite index) do not contain Cartan subalgebras.*

Proof. With the result of J. Feldman and C. C. Moore (Theorem 3.1), every Cartan subalgebra is in particular an abelian subalgebra of multiplicity 1, the statement follows immediately from Corollary 3.1. □

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UCLA MATHEMATICS DEPARTMENT, LOS ANGELES, CA 90095-1555

E-mail address: stefan@math.ucla.edu